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THE FORMULATION AND AN EXISTENCE THEOREM FOR A VARIATIONAL PROBLEM ON PHASE TRANSITIONS IN CONTINUOUS MEDIUM MECHANICS†

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A model of phase transitions in problems of the mechanics of a continuous medium is considered, which goes back to Gibbs [1] and was further developed in [2–5]. An extension of the variational formulation of the problem is proposed, which makes it possible to prove a theorem on the existence of a global maximum of the energy functional for a thermoelastic medium for certain restrictions on the specific energy density.

1. PHYSICAL FORMULATION OF THE PROBLEM

The mathematical feature of the variational problem under consideration is that it is necessary to vary the vector-valued displacement function u simultaneously with the location of phases $\Omega^\pm$. The physical nature of the problem concerning the origin and development of the nucleation centres of a new phase, which have a rather complex structure, makes it necessary to consider the characteristic functions of $\Omega^\pm$ belonging to a special functional space, namely, the space of functions with bounded variation. The above reasons introduce difficulties into the mathematical problem.

An elastic medium occupying a bounded domain $\Omega\subset R^m$, $(m=2, 3)$ is characterized by the displacement field, i.e. a vector-valued function $u(x)$ equal to zero on the boundary of the domain, and by the temperature T, constant over the whole body. It is assumed that the elastic medium can exist in a two-phase state. Each phase is characterized by its deformation energy density $F^\pm(\dot{u}(x), u(x), x, T)$ (the plus and minus superscripts correspond to the first and second phase, respectively, and $\dot{u}(x)\equiv\nabla u(x)$ is the matrix formed by the first-order derivatives of $u(x)$) and its location in the non-deformed state, i.e. by sets $\Omega^\pm$, where $\Omega^+\cap\Omega^-=\phi$ and $\Omega^+\cap\Omega^-=\Omega$. It is natural to take the functional

$$I[u,\Omega^+,T]=\int\limits_{\Omega^+}\rho^+F^+(\dot{u},u,x,T)dx+\int\limits_{\Omega^-}\rho^-F^-(\dot{u},u,x,T)dx, \tag{1.1}$$

$$\Omega^-=\Omega\setminus\Omega^+,\quad u=u(x)$$

as the total deformation energy.

Here $\rho^\pm$ is the density of the medium in the plus and minus phases, respectively. The minimum of (1.1) is to be found among all admissible vector fields $u(x)$ and sets Ω^+. The function $u(x)$ and set Ω^+ that minimize (1.1) define an equilibrium state of the two-phase

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elastic medium. If either Ω^+ or Ω^- is empty, the medium in an equilibrium state consists of a single phase only.

Unfortunately, (1.1) may also not reach its minimum.

To explain this assertion we consider a model example in which $\Omega \subset R^3$ is the spherical layer $0 < r_1 \leq r \leq r_2 < \infty$ and $u(r)$ is a scalar function (a displacement along the radius), $\dot{u}(r)$ being its derivative. When there are radical displacements only the non-zero components of the deformation tensor have the form $e_{rr} = \dot{u} + u/r$ and $e_{\phi\phi} = u/r$ in spherical coordinates.

We define $F^\pm$ by

$$F^\pm = F^\pm(e_{rr}, e_{\phi\phi}) = (e_{rr} \mp 1)^2 + e_{\phi\phi}^2 = (\dot{u} + u/r \pm 1)^2 + (u/r)^2 \tag{1.2}$$

We shall minimize the functional (1.1), (1.2) with $\rho^+ = \rho^- = 1$

$$I[u,\Omega^+] = 4\pi \int_{\Omega^+} r^2[(\dot{u} + u/r - 1)^2 + (u/r)^2]dr + 4\pi \int_{\Omega^-} r^2[(\dot{u} + u/r + 1)^2 + (u/r)^2]dr$$

on the set of functions $u(r) \in \overset{0}{W}{}_2^1\,[r_1, r_2]$ and measurable sets Ω^+. The functional (1.1), (1.2) is positive for every $u \in \overset{0}{W}{}_2^1\,[r_1, r_2]$ and every measurable set Ω^+. However, $\inf I[u, \Omega^+]$ over all $u \in \overset{0}{W}{}_2^1\,[r_1, r_2]$ and all measurable sets Ω^+ in the interval $[r_1, r_2]$ is equal to zero.

In fact, let us divide $[r_1, r_2]$ into 2^n equal intervals. We number the resulting intervals starting from r_1. We take Ω_n^+ to be the sequence consisting of unions of all intervals with odd numbers and Ω_n^- with even ones. Let $u_n(r)$ be a continuous piecewise continuously differentiable function such that $\dot{u}_n(r) = 1$ for $x \in \Omega_n^+$ and $\dot{u}_n(r) - 1$ for $x \in \Omega_n^-$ with $u_n(r_1) = u_n(r_2) = 0$. Then the function $u_n(r)$ has the form of a "saw" and belongs to $\overset{0}{W}{}_2^1\,[r_1, r_2]$. It is seen that $I[u_n, \Omega_n^+] \to 0$ as $n \to \infty$ for the functional (1.1), (1.2).

The example in question is constructed by the method employed in [6]. We note the energy density (1.2) satisfies the material indifference principle [7].

Note that in the above example the minimizing sequence $u_n(r)$, Ω_n^+ for (1.1), (1.2) contains 2^{n-1} phase separation points, the functional I, which can be regarded as the energy, converging to zero.

It is natural to introduce an antagonistic term in (1.1) similar to the Griffith surface energy and proportional to the area of the generated separation surface. The energy functional (1.1) is then replaced by

$$I[u,\Omega^+,T] = \int_{\Omega^+} \rho^+ F^+(\dot{u}, u, x, T)dx + \int_{\Omega^-} \rho^- F^-(\dot{u}, u, x, T)dx + \sigma|S| \tag{1.3}$$

where S is the boundary between Ω^+ and Ω^-, $|S|$ is its area, and σ is a positive constant. The term $\sigma|S|$ has a physical interpretation as the surface energy distributed on the boundary between the phases and caused by the different nature of the medium of each phase.

2. FUNCTIONS OF BOUNDED VARIATION AND CACCIOPPOLI SETS

Because sets of rather complex structure are admissible as Ω^+, the meaning of $|S|$ in (1.3) must be refined and extended. The notion of the surface area of the boundary of a set can be extended by a traditional method used in the theory of minimal surfaces [8] and based on the theory of functions of bounded variation.

Let $\Omega \subset R^m$, $m \geq 2$ be a bounded domain and let $f(x) \in L_1(\Omega)$. We set (in what follows integration is always over Ω)

$$\int|Df| = \sup\{\int f(x)\operatorname{div} g(x)dx\colon\ g \in C_0^1(\Omega, R^m),\ |g(x)| \leq 1 \quad \text{for} \quad x \in \Omega\} \tag{2.1}$$

We shall say that $f \in L_1(\Omega)$ has bounded variation in Ω if $\int |Df| < \infty$.

The above definition of variation is a natural generalization of the one-dimensional case. We recall that a function of one variable has a bounded variation if it can be decomposed into the sum of a continuous function and a step function.

We denote by $BV(\Omega)$ the linear space of functions of bounded variation belonging to $L_1(\Omega)$. The space $BV(\Omega)$ is a Banach space with norm $\| f \|_{BV} = \| f \|_{L_1} + \int | Df |$.

It is obvious that $W_1^1(\Omega) \subset BV(\Omega)$.

As the following example shows, the opposite inclusion is not true. Let $\omega \subset R^m$, let $\partial\omega \in C^2$, and let $\chi(x)$ be the characteristic function of $\omega \cap \Omega$. If $\omega \cap \Omega \neq \Omega$, then $\chi \not\subset W_1^1(\Omega)$. However

$$\int |D\chi| = |(\partial\omega) \cap \Omega| \tag{2.2}$$

where $|(\partial\omega) \cap \Omega|$ is the area of the part of the boundary of ω contained in Ω.

Definition. A measurable set $A \subset \Omega$ is called a Caccioppoli set if its characteristic function $\chi_A(x)$ belongs to $BV(\Omega)$. The finite number $|D\chi_A|$ is called the perimeter of A.

Clearly, $\Omega \setminus A$ is also a Caccioppoli set and the perimeters of A and $\Omega \setminus A$ coincide by virtue of (2.1). We also observe that $\partial\Omega$ does not contribute to the perimeter of A. In the case when A has piecewise-smooth boundary the generalized definition of the area of the boundary surface coincides with the classical one.

Using the above mathematical ideas we can recast the variational problem (1.3). We take as Ω_+ in (1.3) an arbitrary measurable set with finite perimeter and we shall mean by $|S|$ the value of its perimeter, $|S| = \int |D\chi$. where χ is the characteristic function of Ω_+. In terms of the characteristic function χ and its variation the functional (1.3) can be written as

$$I[u, \chi, T] = \int \{\chi\rho^+ F^+(\dot{u}, u, x, T) + (1-\chi)\rho^- F^-(\dot{u}, u, x, T)\} dx + \sigma \int |D\chi| \tag{2.3}$$

The representation (2.3) is a natural extension of (1.3). An advantage of (2.3) is that it has a purely analytic character and contains no geometrical objects, i.e. the sets $\Omega^\pm$ or their separation boundary, which would be inconvenient for later investigation. The representation (2.3) enables us to investigate the variational problem using the whole arsenal of methods for studying functionals in a Banach space. The change from domains $\Omega^\pm$ with smooth boundary to sets with finite perimeter corresponds to "completing" the space of admissible domains.

It should be noted that the need to introduce sets with complex geometry corresponds to the physical nature of the problems concerned with the origin and development of the nucleation centres of a new phase.

In the present paper we state the restrictions on the densities $F^\pm$, for which (2.3) attains its minimum on the set of admissible displacements and study the process giving rise to a two-phase state in the classical theory of thermoelasticity.

Note that the various alternative spaces of functions of bounded variation are applicable in other areas of the mechanics of continuous media, for example, in the theory of plasticity [9].

To conclude the present section we shall state an assertion, which will prove useful later.

Lemma 2.1. $\Omega \subset R^m$ be a bounded domain whose boundary satisfies the Lipschitz condition. Then for all $p \in [l, m/(m-1))$ and all characteristic functions $\chi(x)$ that satisfy the conditions

$$\int \chi dx \leqslant |\Omega|/2, \quad \int |D\chi| < \infty \tag{2.4}$$

the inequality

$$\int \chi dx \leqslant \gamma (\int \chi dx)^{1/p'} \int |D\chi| \tag{2.5}$$

is satisfied, where $1/p' + 1/p = 1$, $|\Omega|$ is the measure of Ω, and $\gamma = \gamma(p, \Omega) > 0$ is a fixed constant.

We will give an outline of the proof of the lemma. Using the theorem on the compactness of the embedding of $W_1^1(\Omega)$ into $L_p(\Omega)$ and Egorov's theorem, one can prove the inequality

$$\left(\int |f|^p dx\right)^{1/p} \leq \gamma(p, \Omega)\int |\nabla f| dx \tag{2.6}$$

which holds for all $f(x) \in W_1^1(\Omega)$ vanishing on a set (depending on the function) of measure not less than $|\Omega|/4$. The inequality (2.6) can be extended to functions from $VB(\Omega)$. This extension has the form

$$\left(\int |f|^p dx\right)^{1/p} \leq \gamma \int |Df|, \quad f \in VB(\Omega), \tag{2.7}$$

where f vanishes in a set (which depends on f) of measure not less than $|\Omega|/2$. If $f = \chi$, where χ is a characteristic function that satisfies the hypothesis of the lemma, then, by (2.7) we get

$$\int \chi dx = \int \chi\chi dx \leq \left(\int \chi^{p'} dx\right)^{1/p'} \left(\int \chi^p dx\right)^{1/p} \leq \gamma \left(\int \chi dx\right)^{1/p'} \int |D\chi|$$

3. THEOREM ON THE EXISTENCE OF A GLOBAL MINIMUM

Let $\Omega \subset R^m$ be a bounded domain with Lipschitz boundary and let $R^{m\times m}$ be the space of $(m\times m)$-matrices. For $D \in R^{m\times m}$, $u \in R^m$, and $x \in \Omega$ we specify $F^{\pm}(D, u, x, T)$. Henceforth, we take a displacement field $u(x)$ as u and the matrix formed by the first-order derivatives of $u(x)$ as D. The derivatives with respect to the corresponding argument will be denoted by subscripts attached to these functions. We denote by $|\cdot|$ both the modulus of a scalar or vector-valued function, and the norm of a matrix. We fix numbers p and α and a function $C(T)$ such that

$$p \in (1, \infty), \quad \alpha > 0, \quad C(T) > 0$$

We assume that $F^{\pm}$ are jointly continuous functions with respect to their arguments, continuously differentiable and convex with respect to the components of D, and satisfy the inequalities

$$\begin{aligned} &|F^{\pm}(D, u, x, T)| \leq C(T)[|D|^p + |u|^p + 1], \quad |F_D^{\pm}(D, u, x, T)| \leq C(T)[|D|^{p-1} + |u|^{p-1} + 1] \\ &C^{-1}(T)[|D|^p - |u|^s - \alpha] \leq F^{\pm}(D, u, x, T), \quad s \in [1, p) \end{aligned} \tag{3.1}$$

For the characteristic function $p \in (1, \infty)$, $\alpha > 0$, $C(T) > 0$ of a Caccioppoli set $\Omega^+ \subset \Omega$, a vector-valued function $u(x) = \overset{0}{W}{}_p^1(\Omega, R^m)$, and a fixed value $T \in R^1$ we define $I(u, \chi, T)$ by (2.3). This functional is well defined. It will be called the energy functional of a two-phase medium.

Finally, let us state the variational problem: among all functions

$$u(x) = \overset{0}{W}{}_p^1(\Omega, R^m), \quad \chi(x) \in BV(\Omega) \tag{3.2}$$

($\chi(x)$ being the characteristic function) it is required to find those that realize the minimum of $I(u, \chi, T)$ for a fixed T.

Existence theorem. The functional $I(u, \chi, T)$ attains its global minimum in the set (3.2) for every fixed T.

The proof of this theorem, even under more general assumptions, can be found in [10].

4. THE PROBLEM OF PHASE TRANSITIONS IN THE CLASSICAL THEORY OF ELASTICITY

The above variational formulation enables us to invoke the direct methods of variational calculus to answer the question of the qualitative behaviour of a medium in a two-phase state. We shall illustrate these possibilities in the case of the classical theory of elasticity. We assume that the phases differ only by their coefficients of thermal expansion $\alpha^{\pm}>0$, having the same Lamé coefficients $\lambda>0$ and $\mu>0$. In this case the densities $F^{\pm}$ are given by [11]

$$F^{\pm}(\dot{u},T')=-\left(\lambda+\frac{2\mu}{3}\right)\alpha^{\pm}T\operatorname{div}u+\frac{\lambda}{2}(\operatorname{div}u)^2+\frac{\mu}{2}[u^i_{x_j}u^i_{x_j}+u^i_{x_j}u^j_{x_i}]+F^{\pm}(0,T_0)=$$
$$=-\left(\lambda+\frac{2\mu}{3}\right)\alpha^{\pm}TE_{\rm I}+\frac{\lambda+2\mu}{2}E_{\rm I}^2-2\mu E_{\rm II}+F^{\pm}(0,T_0) \tag{4.1}$$

where $E_{\rm I}$ and $E_{\rm II}$ are the first and second invariants of the deformation tensor $T=T'-T_0$ and, for $\rho^+=\rho^-=1$ and $F^+(0,\ T_0)=F^-(0,\ T_0)$ the functional (2.3) can be represented in the form

$$I[u,\chi,T]=\int\left\{\frac{\lambda+\mu}{2}(\operatorname{div}u)^2+\frac{\mu}{2}u^i_{x_j}u^i_{x_j}\right\}dx+\sigma\int|D\chi|-$$
$$-\left(\lambda+\frac{2\mu}{3}\right)T[\alpha]\int\chi\operatorname{div}u\,dx,\quad [\alpha]=\alpha^+-\alpha^-\neq 0 \tag{4.2}$$

or, on extracting a perfect square, as

$$I[u,\chi,T]=\int\left\{\frac{\lambda+\mu}{2}(\operatorname{div}u-\xi T[\alpha]\chi)^2+\frac{\mu}{2}u^i_{x_j}u^i_{x_j}\right\}dx+$$
$$+\sigma\int|D\chi|-\frac{\xi}{2}T^2[\alpha]^2\int\chi dx,\quad \xi=\frac{\lambda+2\mu/3}{\lambda+\mu} \tag{4.3}$$

It follows that the phase transition problem is to be studied near the temperature T_0, the phases being indistinguishable when $T'=T_0$.

The following mathematical results have physical meaning for a finite variation of T near zero. For $p=2$ it is obvious that the hypotheses of the existence theorem for the densities (4.1) are satisfied. It follows from (4.2) that the following equalities hold for u and χ from the set (3.2)

$$I[u,\chi,T]=I[\hat{u},\hat{\chi},T],\quad \hat{u}=-u,\quad \hat{\chi}=1-\chi;\quad I[u,\chi,T]=I[-u,\chi,-T] \tag{4.4}$$

$\hat{u}$ and $\hat{\chi}$ also being members of (3.2). Therefore the family of pairs $\{u,\chi\}$ that minimize (2.3) on the set (3.2) consists of at least two pairs. Since $\hat{\chi}=1-\chi$, one of the integrals

$$\int\chi dx\quad\text{or}\quad\int\hat{\chi}dx \tag{4.5}$$

is not greater than a half of the measure of Ω.

For $T=0$ the functional (2.3) attains its last value on the set (3.2) only for $\{u,\chi\}$ with $u\equiv 0$ and $\chi\equiv 0$ or $\chi\equiv 1$, which corresponds to the single-phase state of an elastic medium. The purpose of the present section is to study the origin of the two-phase state as the temperature changes.

For each temperature $T\in R^1$ we define

$$\hat{I}[T]=\inf I[u,\chi,T]$$

where the infimum is taken over all pairs $\{u,\chi\}$ from (3.2). Note that $\hat{I}[T]\leqslant I[0,0,\ T]=0$.

Lemma 4.1. A number k, $0<k<\infty$ exists such that

$$\hat{I}[T]=0\quad\text{for}\quad |T|\leqslant k,\quad \hat{I}[T]<0\quad\text{for}\quad |T|>k$$

Proof. We define the following sets M and N on the real axis

$$M=\{T\in R^1,\ \hat{I}[T]<0\},\quad N=\{T\in R^1,\ \hat{I}[T]=0\}$$

By the second equality in (4.4), T and $-T$ belong simultaneously either to M or to N. We divide the proof into a number of stages.

1. N *is a non-empty closed set.* Suppose that $\{u,\chi\}$ realizes the minimum of (2.3) on the set (3.2). We shall prove that for $|T|$ small enough the function $\chi(x)$ is either identically equal to one or zero. All T with sufficiently small modulus will then belong to N. The condition $\chi\equiv 0$ or $\chi\equiv 1$ is equivalent to

$$\int|D\chi|=0 \tag{4.6}$$

Since, for a fixed T, the pair $\{\hat{u},\ \hat{\chi}\}(\hat{u}=-u,\ \hat{\chi}=1-\chi)$ also minimizes (2.3) on the set (3.2), we have

$$I[u,\chi,T]=I[\hat{u},\hat{\chi},T]\leqslant I[0,0,T]=0$$

The latter relationship and the representation (4.3) of $I[u,\chi,T]$ lead to the inequality

$$\int|D\chi_0|\leqslant\zeta T^2[\alpha]^2\int\chi_0 dx,\quad \zeta=(\lambda+2\mu/3)^2/[2\sigma(\lambda+\mu)] \tag{4.7}$$

for $\chi_0\equiv\chi$ and $\chi_0\equiv\hat{\chi}$.

Let χ_0 be identical with the function (χ or $\hat{\chi}$) for which

$$\int\chi_0\,dx\leqslant|\Omega|/2 \tag{4.8}$$

Combining (4.7) with (2.5) and taking (4.8) into account, we get

$$\int|D\chi_0|\leqslant\zeta\gamma T^2[\alpha]^2\left(\int\chi_0 dx\right)^{1/p'}\int|D\chi_0| \tag{4.9}$$

Since χ_0 satisfies (4.8), if

$$\zeta\gamma T^2[\alpha]^2(|\Omega|/2)^{1/p'}<1 \tag{4.10}$$

then (4.9) implies that

$$\int|D\chi_0|=0 \tag{4.11}$$

If $\chi_0=\chi$, then (4.11) is identical with (4.6). If $\chi_0=\hat{\chi}$, then (4.6) follows from (4.11) and the equality $\int|D\hat{\chi}|=\int|D\chi|$.

We shall now prove that N is a closed set. Let $T_n\in N$, $(n=1,2,\ldots,)$ $T_n\to T_*$ as $n\to\infty$. We must prove that $\hat{I}[T_*]=0$. Suppose that this is not the case. Let $\hat{I}[T_*]<0$. The functional $I[u,\chi,T_*]$ attains its minimum on the set (3.2) for a certain pair $\{u_0,\ \chi_0\}$. Then $I[u_0,\chi_0,T_*]=\hat{I}[T]<0$. Since $I[u_0,\chi_0,T]$ is a continuous function of T, we arrive at $\hat{I}[T_n]\leqslant I[u_0,\chi_0,T_n]<0$ for sufficiently large n, which is a contradiction.

2. *The set* M *is non-empty. If* $T_*\in M$ *then every* T *such that* $|T|\geqslant|T_*|$ *also belongs to* M. We will first prove that M is non-empty. We fix a non-solenoidal vector field $u(x)\in \overset{0}{W}{}_2^1(\Omega,\ R^m)$. Let $\operatorname{div}u>0$ on a subdomain $\omega\subset\Omega$ with smooth boundary. As χ we take the characteristic function of ω. Then, for sufficiently large T, the functional (2.3) will be negative for the selected u and χ. Therefore $T\in M$ if T is large enough.

We now prove the second part of the assertion. Since $T=0$ belongs to N, it follows that $T_*\neq 0$. Let $T_*>0$, $T_*\in M$ and let $I[u,\chi,\pm T_*]$ attain its minimum on the set (3.2) on the pair $\{\pm u_0,\ \chi_0\}$, i.e.

$$I[\pm u_0,\chi_0,\pm T_*]=\hat{I}[+T_*]=\hat{I}[-T_*]<0$$

The latter relationship and (4.2) imply that

$$[\alpha]\int\chi_0 \mathrm{div}\, u_0 dx > 0 \tag{4.12}$$

Thus

$$\begin{aligned} &I[u_0, \chi_0, T] < I[u_0, \chi_0, T_*] < 0 \quad \text{for} \quad T > T_* \\ &I[-u_0, \chi_0, -T] < I[-u_0, \chi_0, -T_*] < 0 \quad \text{for} \quad T < -T_* \\ &\hat{I}[T] < \hat{I}[T_*] = \hat{I}[-T_*] < 0 \quad \text{for} \ |T| > T_*. \end{aligned} \tag{4.13}$$

The lemma follows from assertions 1 and 2.

Lemma 4.2. For $|T| < k$ the functional (2.3) attains its minimum on the set (3.2) only for $\{u, \chi\}$ with $u \equiv 0$ and $\chi \equiv 0$ or $\chi \equiv 1$. When $|T| > k$ the functional (2.3) attains its minimum on (3.2) only for $\{u, \chi\}$ such that $u \not\equiv 0$ $\chi \not\equiv 0$ and $\chi \not\equiv 1$.

Proof. For $|T_*| < k$ let the minimum be attained at $\{u_0, \chi_0\}$ with $\chi_0 \not\equiv 0$ and $\chi_0 \not\equiv 1$. We assume that $T_* > 0$, since for $T_* = 0$ the minimum is realized only by pairs with $u_0 \equiv 0$ and $\chi_0 \equiv 0$ or $\chi_0 \equiv 1$, and (by the second inequality in (4.4)) replacing T_* by $-T_*$ results just in u_0 being replaced by $-u_0$. Since $I[u_0, \chi_0, T_*] = \hat{I}[T_*] = 0$, (4.2) implies (4.12), from which, as before, we obtain (4.13), which contradicts the definition of k and the choice of T_*. The resulting contradiction proves that either $\chi_0 \equiv 0$ or $\chi_0 \equiv 1$. In this case $u_0 \equiv 0$.

Since for $\chi \equiv 0$ and $\chi \equiv 1$ the functional (4.2) is non-negative, it follows that for $|T_*| > k$ the minimum is realized by $\{u_0, \chi_0\}$ such that $\chi_0 \not\equiv 0$, $\chi_0 \not\equiv 1$, and $u_0 \not\equiv 0$.

Lemma 4.3. Let $|T_*| > k$ and let the minimum of (2.3) on the set (3.2) be realized by $\{u_0, \chi_0\}$ for $T = T_*$. Then

$$1 < \zeta\gamma T_*^2 [\alpha]^2 \left(\int\chi_0 dx\right)^{1/p'} \tag{4.14}$$

where γ is the constant in Lemma 2.1, $1/p + 1/p' = 1$, and $1 < p < m/(m-1)$.

Proof. Replacing u_0 by $-u_0$ and χ_0 by $1-\chi_0$ if necessary, we shall assume that χ_0 satisfies (4.8) Using (4.7) (which holds for χ_0 from the minimizing pair), the estimate from Lemma 2.1 (valid by virtue of (4.8)) and cancelling the non-vanishing factor $\int |D\chi_0|$, (by virtue of Lemma 4.2 and the assumption that $|T_*| > k$), we obtain (4.14).

Let us consider the physical interpretation of Lemmas 4.1–4.3. Since only single-phase states with vanishing displacement field are possible for $|T| < k$, they give rise to two isolated minima of the energy functional. Suppose that a state with density F^- (i.e. $\chi \equiv 0$) is realized for $T = 0$. This state is then preserved for temperatures $T \in (-k, k)$. A two-phase state ($\chi \not\equiv 0$, $\chi \not\equiv 1$) with non-zero displacement field appears when the temperature crosses the critical value $^{\pm}k$, the occurrence of the phase with density F^+ having a jump-like character (inequality (4.14)).

Consider the single-phase state $u \equiv 0$, $\chi \equiv 0$ for $|T| > k$. We shall prove that it remains a local minimum of the energy functional in the following sense.

Lemma 4.4. For every T a positive number $\delta = \delta(T) \leqslant |\Omega|/2$ exists such that for all $u \in \overset{0}{W}{}_2^1(\Omega, R^m)$ and all characteristic functions $\chi(x) \in BV(\Omega)$ with

$$0 < \int\chi(x)dx \leqslant \delta(T) \tag{4.15}$$

the inequality

$$I[u, \chi, T] > I[0, 0, T] = 0. \tag{4.16}$$

is satisfied.

Proof. Suppose that the opposite inequality to (4.16) is satisfied for some u_0 and χ_0 with $\chi_0 \not\equiv 0$ and $\int \chi_0 dx \leqslant |\Omega|/2$. Then (4.7) and, consequently, (4.9) are satisfied for this pair. If

$$\zeta \gamma T^2 [\alpha]^2 (\int \chi_0 dx)^{1/p'} < 1 \tag{4.17}$$

then χ_0 satisfied (4.11), which contradicts the first inequality in (4.15). We determine $\delta(T)$ from (4.17).

The latter two lemmas show that the origin of a new phase resembles the process of "flipping" in the theory of shells.

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